Динамика невискозних флуида

* Напонско стање

– Нема тангенц. напона: $\sigma_{ij}=0,\ i\neq j$

– Вредност нормалног напона у једној тачки једнака без обзира на оријентацију површи у којој.

$$\begin{bmatrix}\sigma_{xx} & 0 & 0\\ 0 & \sigma_{yy} & 0\\ 0 & 0 & \sigma_{zz}\end{bmatrix}=\begin{bmatrix}-p & 0 & 0\\ 0 & -p & 0\\ 0 & 0 & -p\end{bmatrix} \qquad \vec{\sigma}_n=-p\vec{n}$$

* Лагранж

$$\frac{D\vec{K}}{Dt}=\vec{F}_m+\vec{F}_n \Big/ K=mU=\rho V U \rightarrow \frac{D}{Dt}\iiint_V \rho\vec{U}\,dV=\iiint_V \vec{f}\rho\,dV+\oint\!\!\!\oint_A -p\vec{n}\,dA$$

$$\boxed{\iiint\left(\frac{D\vec{U}}{Dt}\rho-\vec{f}\rho+\mathrm{grad}\,p\right)dV=0} \rightarrow \boxed{\frac{D\vec{U}}{Dt}\rho=\vec{f}\rho-\mathrm{grad}\,p} \Big/ \vec{i} \qquad x:\ \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}$$

↳ ИНТЕГРАЛНИ ОБЛИК ↳ ДНФ.

– природне координате:

$$s:\ \boxed{\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial s}u\right)=f_s\cdot\rho-\frac{\partial p}{\partial s}} \qquad n:\ \boxed{\rho\frac{U^2}{R_k}=\rho f_n-\frac{\partial p}{\partial n}}$$

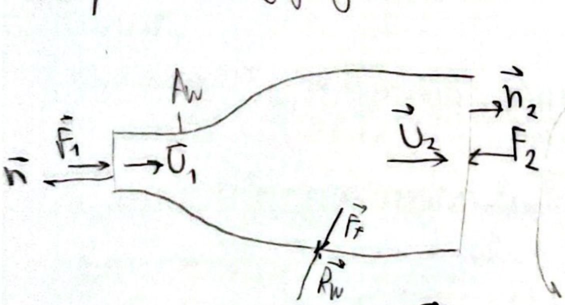
* Ојлер

$$\frac{d\vec{K}}{dt}=\vec{K}_u-\vec{K}_i+\vec{F}_m+\vec{F}_n \Rightarrow \frac{D}{Dt}\iiint\rho\vec{U}\,dV=-\underbrace{\iint_{A_u}\rho\vec{U}(\vec{U}\cdot\vec{n})\,dA}_{0}-\iint_{A_i}\rho\vec{U}(\vec{U}\cdot\vec{n})\,dA+\iiint_V\rho\vec{f}\,dV-\oint\!\!\!\oint_A\vec{n}p\,dA$$

Рејнолдс: $\frac{D}{Dt}\iiint f\,dV=\iiint\frac{\partial f}{\partial t}\,dV+\oint\!\!\!\oint f(\vec{U}\cdot\vec{n})\,dA\ ,\quad \vec{f}=\rho\vec{U}$

$$↳\ \boxed{\frac{D}{Dt}\iiint\rho\vec{U}\,dV=\iiint\frac{\partial(\rho\vec{U})}{\partial t}\,dV+\oint\!\!\!\oint\rho\vec{U}(\vec{U}\cdot\vec{n})\,dA=\iiint\rho\vec{f}\,dV+\iint_A -p\vec{n}\,dA}$$

* Примена на једнодимензијски модел



$$\oint\!\!\!\oint_A\rho\vec{U}(\vec{U}\cdot\vec{n})\,dA=\iiint\rho\vec{f}\,dV+\oint\!\!\!\oint_A -p\vec{n}\,dA$$

$$\oint\!\!\!\oint_A\rho\vec{U}(\vec{U}\cdot\vec{n})\,dA=\underbrace{\iint_{A_1}\rho\vec{U}(\vec{U}\cdot\vec{n})\,dA}_{①}+\underbrace{\iint_{A_2}\dots}_{②}+\underbrace{\iint_{A_w}\dots}_{③}=$$

$$=\iiint\rho\vec{f}\,dV+\underbrace{\iint_{A_1}-p\vec{n}\,dA}_{\vec{F}_1}+\underbrace{\iint_{A_2}-p\vec{n}\,dA}_{\vec{F}_2}+\underbrace{\iint_{A_w}-p\vec{n}\,dA}_{R_w}$$

$\measuredangle(\vec{U}_1,\vec{n}_1)=\pi \rightarrow ① \rightarrow -\rho U_1 A_1\vec{U}_1$

$\measuredangle(\vec{U}_2,\vec{n}_2)=0 \rightarrow ② \rightarrow \rho_2 U_2 A_2\vec{U}_2$

$\measuredangle(U_w,\vec{n}_w)=\frac{\pi}{2} \rightarrow ③ \rightarrow 0$

$$-\dot m_1 \vec U_1 + \dot m_2 \vec U_2 = \iiint \rho \vec f \, dV + \vec F_1 + \vec F_2 + \vec R_W \quad / \quad \vec R_W = -\vec F_{fe}$$

$$\boxed{\vec F_{fe} = \rho V \vec g + \dot m_1 \vec U_1 - \dot m_2 \vec U_2 + \vec F_1 + \vec F_2}$$

* Бернулијев интеграл

$$\boxed{\frac{d\vec U}{dt} + \frac{1}{2}\operatorname{grad} U^2 - 2\vec U \times \vec \omega = \vec f - \frac{1}{\rho}\operatorname{grad} p}$$

• Претпоставке

1. Стационарно струјање → $\frac{\partial \vec U}{\partial t} = 0$
2. Запр. силе су конзервативне → $\vec f = \operatorname{grad}\Phi$, $\Phi = -gz$
3. Флуид је баротропан → $\frac{1}{\rho}\operatorname{grad} p = \operatorname{grad}\int\frac{dp}{\rho(p)}$; $\rho = \rho(p)$

↓ + претпоставке

$$\boxed{\operatorname{grad}\left(\int\frac{dp}{\rho(p)} - \Phi + \frac{1}{2}U^2\right) = 2\vec U \times \vec \omega}$$

$\vec U \times \vec \omega = 0$: 1. $\vec U = 0$
2. $\operatorname{rot}\vec U = 0 \to \vec\omega = 0$
3. $\vec U \parallel \vec \omega \to 0$

↓ + услови

$$\boxed{\int\frac{dp}{\rho(p)} - \Phi + \frac{1}{2}U^2 = const}$$

– Јединична механичка енергија је иста у свим тачкама струјног поља ако је тачно: – невртложно – $\vec U \parallel \vec\omega$

– Ако је струјање невртложно функција има потенцијал ($\operatorname{rot}(\operatorname{grad}\varphi) = 0$), а ако је и нестишљиво $\operatorname{div}\vec U = 0$, $\operatorname{div}(\operatorname{grad}\varphi) = 0$, па потенцијал функције задовољава Лапласову ј-ну

$$\boxed{\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \Delta\varphi = 0}$$

– Бернулијева ј-на се добија множењем са $d\vec s$:

$$\operatorname{grad}\left(\int\frac{dp}{\rho(p)} - \Phi + \frac{1}{2}U^2\right)\cdot d\vec s = 2\underbrace{(\vec U \times \vec\omega)\cdot d\vec s}_{0}$$

$$\boxed{\int\frac{dp}{\rho(p)} - \Phi + \frac{1}{2}U^2 = C_\psi}$$

– Јед. мех. енергија се не мења дуж струјнице!

$\int\frac{dp}{\rho(p)}$ – струјна енергија

$\frac{U^2}{2}$ – кинетичка енергија

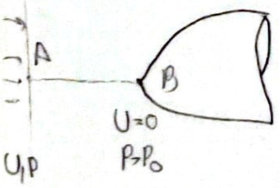
$-\Phi$ – потенцијална

* Бернули за нестишљив флуид

$$\boxed{\frac{p}{\rho} + \frac{U^2}{2} + gz = const.}$$

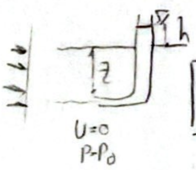
– Када су струјнице праве линије притисак се у правцу нормале на њих расподељује као у хидростатици

$$\boxed{p_0 = p + \frac{\rho U^2}{2}}$$



p_0 – зауставни притисак

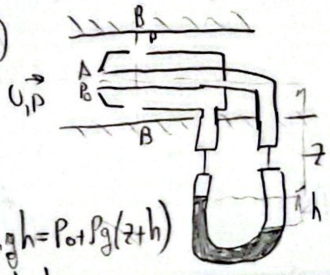
* Пито сонда



$$p_a + \rho g z + \frac{\rho U^2}{2} = p_a + \rho g(h+z)$$

$$\boxed{U = \sqrt{2gh}}$$

* Пито-Прандтлова цев



$$p + \frac{\rho U^2}{2} = p_0 \quad (1)$$

$$\boxed{U = \sqrt{\frac{2(p_0 - p)}{\rho}}}$$

(2)

$$p + \rho g z + \rho_m g h = p_0 + \rho g(z+h)$$

$$p_0 - p = (\rho_m - \rho) g h \longrightarrow \boxed{U = \sqrt{\frac{2(\rho_m - \rho) g h}{\rho}}}$$

* Кавитација

- Када падне притисак доведе до кључања и појаве двофазно струјање → $\boxed{p > p_{isp.}}$

* Бернулијева ј-на за стишљив флуид

$$\boxed{\int \frac{dp}{\rho(p)} + \frac{U^2}{2} = Const.}$$

- Изотермско: $\rho = \frac{p}{RT} \rightarrow \boxed{RT\ln p + \frac{U^2}{2} = const.}$
- Адијабатско

* Загревање тела

$$\boxed{T_0 = T + \frac{U^2}{2c_p}} \quad \boxed{c_p = \frac{kR}{k-1}}$$

- Потпуне величине стања: $\boxed{U=0}$
- Критичне величине стања: $\boxed{U=M}$

* Брзина звука

$$\boxed{C = \sqrt{kRT}} \quad \boxed{M = \frac{U}{C}}$$

Динамика вискозног флуида

* Напонско стање

$$\tilde{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$\sigma_{xx} = -p + \tau_{xx}$

$\sigma_{yy} = -p + \tau_{yy}$

$\sigma_{zz} = -p + \tau_{zz}$

* Навије Стоксова ј-на

• Извод:

$$\frac{D\vec{K}}{Dt} = \vec{F}_m + \vec{F}_n \rightarrow \frac{D}{Dt}\iiint \rho\vec{U}dV = \iiint \rho\vec{f}dV + \oint\!\!\oint_A -p\vec{n}dA$$

$$\oint\!\!\oint \sigma_n dA = \iiint\left(\frac{\partial\vec{\sigma}_x}{\partial x} + \frac{\partial\vec{\sigma}_y}{\partial y} + \frac{\partial\vec{\sigma}_z}{\partial z}\right)dV$$

$$\boxed{\iiint\left(\rho\frac{D\vec{U}}{Dt} - \rho\vec{f} - \frac{\partial\vec{\sigma}_x}{\partial x} - \frac{\partial\vec{\sigma}_y}{\partial y} - \frac{\partial\vec{\sigma}_z}{\partial z}\right)dV = 0} \rightarrow \boxed{\rho\frac{D\vec{U}}{Dt} = \rho\vec{f} + \frac{\partial\vec{\sigma}_x}{\partial x} + \frac{\partial\vec{\sigma}_y}{\partial y} + \frac{\partial\vec{\sigma}_z}{\partial z}}$$

↳ ИНТ. ↳ ДНФ.

$$(x):\ \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \rho f_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}$$

- Стоксова хипотеза представља линеарну везу напона у флуиду и брзину деф. $\tilde{\tau} - \tilde{S}$ (Н. Ф.), а компоненте тензора изражавају се у ф-ји поља брзине

$$\boxed{\tau_{ik} = 2\eta S_{ik} + \underbrace{\eta'\delta_{ik}\,div\vec{U}}_{\rho\neq const}}$$

$\rho = const$

η': $\tau_{xx} = 2\eta\frac{\partial u}{\partial x} + \eta' div\vec{U}$, $\tau_{yy} = 2\eta\frac{\partial v}{\partial y} + \eta' div\vec{U}$, $\tau_{zz} = 2\eta\frac{\partial w}{\partial z} + \eta' div\vec{U}$

• Нестишљив: $\rho = const$, $div\vec{U} = 0$

$\tau_{xx} + \tau_{yy} + \tau_{zz} = 2\,div\vec{U} = 0 \rightarrow \boxed{\eta' = 0}$

• Стишљив: $div\vec{U} \neq 0$

$\tau_{xx} + \tau_{yy} + \tau_{zz} = 2\eta\,div\vec{U} + 3\eta' div\vec{U} \rightarrow \boxed{\eta' = -\frac{2}{3}\eta}$

$$S_{ik} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right) \qquad \delta_{ik} = \begin{cases} 1 & i=k \\ 0 & i\neq k \end{cases}$$

Тако су нађени:

$$\tau_{xx}=2\eta\frac{\partial u}{\partial x}-\frac{2}{3}\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \qquad \tau_{xy}=\tau_{yx}=\eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\tau_{yy}=2\eta\frac{\partial v}{\partial y}-\frac{2}{3}\eta \quad -\text{II}- \qquad \tau_{zx}=\tau_{xz}=\eta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\tau_{zz}=2\eta\frac{\partial w}{\partial z}-\frac{2}{3}\eta \quad -\text{II}- \qquad \tau_{yz}=\tau_{zy}=\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\boxed{\rho\frac{D\vec{U}}{Dt}=\rho\vec{f}-\operatorname{grad}P+\eta\Delta\vec{U}+\frac{1}{3}\eta\operatorname{grad}(\operatorname{div}\vec{U})} \;,\quad \boxed{\rho\frac{D\vec{U}}{Dt}=\rho\vec{f}-\operatorname{grad}P+\eta\Delta\vec{U}}$$

↳ Навије-Стокс за стишљив флуид ↳ нестишљив

* Ојлер

$$\frac{\partial\vec{K}}{\partial t}=\vec{K}_u-\vec{K}_i+\vec{F}_m+\vec{F}_n \quad \Big/ -\iint_{A_u} - \iint_{A_i} = -\oint_A$$

$$\frac{d}{dt}\iiint_V\rho\vec{U}dV=-\oint_A\rho\vec{U}(\vec{U}\cdot\vec{n})dA+\iiint_V\rho\vec{f}dV+\iint_A\vec{\sigma}_n dA \quad \Big/ \frac{D}{Dt}\iiint f dV=\iiint\frac{\partial f}{\partial t}dV+\oint f(\vec{U}\cdot\vec{n})dA,\; \vec{F}=\rho\vec{U}$$

$$\boxed{\frac{D}{Dt}\iiint\rho\vec{U}dV=\iiint_V\frac{\partial(\rho\vec{U})}{\partial t}dV+\oint\rho\vec{U}(\vec{U}\cdot\vec{n})dA=\iiint\rho\vec{f}dV+\iint\vec{\sigma}_n dA}$$

* Бернулијева једначина за вискозно струјање

$$\boxed{\frac{\partial\vec{U}}{\partial t}+\frac{1}{2}\operatorname{grad}U^2-2\vec{U}\times\vec{\omega}=\vec{f}-\frac{1}{\rho}\operatorname{grad}P+\nu\Delta\vec{U}+\frac{1}{3}\nu\operatorname{grad}(\operatorname{div}\vec{U})}$$

• Претпоставке
1. Стационарно струјање $\frac{\partial\vec{U}}{\partial t}=0$
2. Конз. силе → $\vec{f}=\operatorname{grad}\Phi$, $\Phi=-gz$
3. Нестишљив ф. → $\operatorname{div}\vec{U}=0$

↓ + претпоставке

$$\boxed{\operatorname{grad}\left(\frac{P}{\rho}+\frac{U^2}{2}+gz\right)-\nu\Delta\vec{U}=2\vec{U}\times\vec{\omega}} \quad \Big/\cdot d\vec{s},\ \int_1^2$$

↓

$$\int_1^2 dY_s-\int_1^2\nu\Delta\vec{U}d\vec{s}=0 \rightarrow Y_{s_2}-Y_{s_1}+Y_{g_{1-2}} \Rightarrow \boxed{Y_{s_1}=Y_{s_2}+Y_{g_{1-2}}} \quad \Big/\cdot d\dot{m}=\rho U dA$$

↓

$$\iint_{A_1}\left(\frac{P_1}{\rho_1}+\frac{U_1^2}{2}+gz_1\right)\rho U dA=\iint_{A_2}\left(\frac{P_2}{\rho_2}+\frac{U_2^2}{2}+gz_2\right)\rho U dA+Y_{g_{1-2}}d\dot{m} \quad \Big/ \frac{P}{\rho}+gz=\text{const.}$$

$$\iint_A\ldots=\left(\frac{P}{\rho}+gz\right)\underbrace{\iint\rho U dA}_{\dot{m}}+\frac{\rho}{2}\iint U^3 dA$$

$$\iint U^3 dA\neq U^3 A$$

$$\alpha=\frac{\iint U^3 dA}{U^3 A}\begin{cases}=1 - \text{турбулентно}\\=2 - \text{ламинарно}\end{cases}$$

$$\rightarrow \boxed{\frac{P_1}{\rho}+\alpha_1\frac{U_1^2}{2}+gz_1=\frac{P_2}{\rho}+\alpha_2\frac{U_2^2}{2}+gz_2+Y_{\Phi_{1-2}}}$$

Јединична струјна ен. ф. у пресеку 1 једнака је збиру јед. стр. ен. у пресеку 2 и јед. губитка енергије услед вискозности

* Губици енергије

$$\boxed{Y_{g\,1-2} = Y_{\lambda\,1-2} + Y_{\xi\,1-2}}$$

праволин. — локални отпори

• праволинијске деонице

$\boxed{Y_\lambda = f l \frac{O}{A} \frac{U^2}{2}}$ $f = \frac{\lambda}{4}$, O - оквашени део $\boxed{\tau_w = f \frac{\rho U^2}{2}}$ $\boxed{Y_\lambda = \lambda \frac{L}{D_h} \frac{U^2}{2}}$

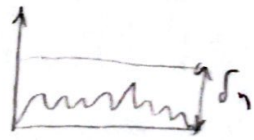
↳ $\boxed{D_h = 4A/O}$

λ - коеф. трења $\boxed{\lambda = \frac{64}{Re}}$, зависи од $Re = \frac{UD}{\nu}$ и $K = \frac{\delta}{D}$

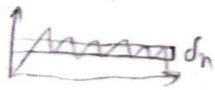
! Re - однос инерцијалних и вискозних сила

– Мудијев дијаграм - зависност λ-Re

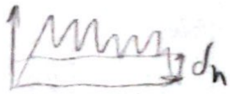
1. $Re \leq 2320$: ламинарно струјање, $\boxed{\lambda = \frac{64}{Re}}$
2. $2320 \leq Re \leq 4000$: Прелазна област, $\lambda = 0{,}0025\sqrt[3]{Re}$
3. $Re > 4000$: турбулентно
   - 3.1. $Re < \frac{40}{K}$: хидраулички глатка



   - 3.2. $\frac{40}{K} < Re < \frac{500}{K}$: хидраулички храпава



   - 3.3. $\frac{500}{K} < Re$: потпуно храпава



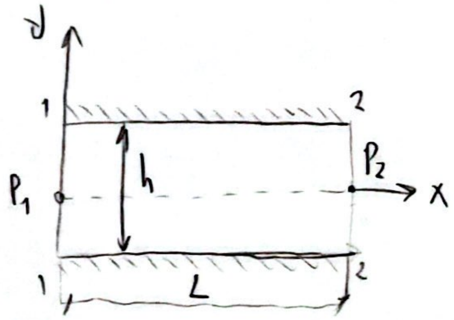
• Локални губици

$$\boxed{Y_\xi = \xi \frac{U^2}{2}}$$

• ширење $\boxed{Y_p = \frac{(U_{ve} - U_{nc})^2}{2} = \frac{U_v^2}{2}}$ • рачва

Тачна решења

* Стационарно, нестишљиво струјање између паралелних плоча



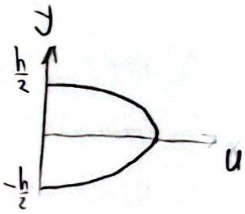
Претпоставке:
- стационарно: $\frac{\partial()}{\partial t}=0$
- изотермско: $T=const.$
- нестишљиво: $\rho=const$
- раванско: $\frac{\partial()}{\partial z}=0,\ w=0$
- масене силе се занемарују: $f=0$

$$\boxed{\frac{dp}{dx}=\eta\frac{d^2u}{dy^2}=-k}$$

• брзина

$$\frac{d^2u}{dy^2}=-\frac{k}{\eta}\Big/\int,\quad \frac{du}{dy}=-\frac{k}{\eta}y+C_1\Big/\int,\quad \boxed{u=-\frac{k}{\eta}\frac{y^2}{2}+C_1y+C_2}$$

$$\left.\begin{aligned} y=\frac{h}{2},\ u=0 &\rightarrow 0=-\frac{k}{\eta}\frac{h^2}{8}+C_1\frac{h}{2}+C_2 \\ y=-\frac{h}{2},\ u=0 &\rightarrow 0=-\frac{k}{\eta}\frac{h^2}{8}-\frac{C_1}{2}h+C_2 \end{aligned}\right\}\begin{aligned} &C_1=0 \\ &C_2=\frac{k}{\eta}\frac{h^2}{8}\end{aligned} \rightarrow \boxed{u=-\frac{k}{\eta}\frac{y^2}{2}+\frac{k}{\eta}\frac{h^2}{8}=\frac{kh^2}{2\eta}\left(\frac{1}{4}-\frac{y^2}{h^2}\right)}$$



• максимална брзина

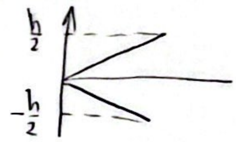
$$y=0\rightarrow \boxed{u=\frac{kh^2}{8\eta}}$$

• проток и средња брзина

$$\dot V=2\int_0^{\frac{h}{2}}u\,dy=2\int_0^{\frac{h}{2}}\frac{kh^2}{2\eta}\left(\frac{1}{4}-\frac{y^2}{h^2}\right)dy=\boxed{\frac{kh^3}{12\eta}}\qquad u_s=\frac{\dot V}{h}\rightarrow\boxed{u_s=\frac{kh^2}{12\eta}}$$

• тангенцијални напон

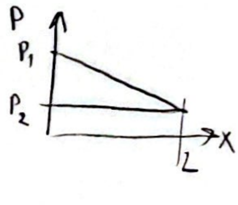
$$\tau_w=-\eta\left.\frac{du}{dy}\right|_{\pm\frac{h}{2}}=\eta\left(0-\frac{k}{\eta}y\right)=\boxed{\frac{kh}{2}}$$



• расподела притиска

$$\frac{dp}{dx}=-k,\quad dp=-k\,dx,\quad \boxed{p=-kx+C}$$

$$\left.\begin{aligned} x=0,\ p=p_1 &\rightarrow p_1=C \\ x=L,\ p=p_2 &\rightarrow p_2=-kL+C \end{aligned}\right\} p_1=p_2+kL,\ \boxed{k=\frac{\Delta p}{L}}\ \Bigg\}\ \boxed{p=p_1-\frac{\Delta p}{L}x}$$

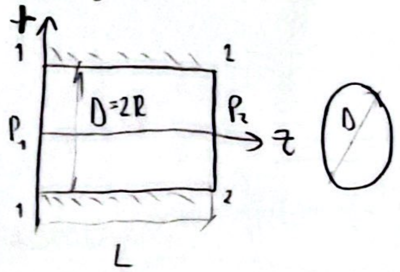


• коеф. трења

$$\frac{p_1}{\rho}+\frac{u_{1s}^2}{2}=\frac{p_2}{\rho}+\frac{u_{2s}^2}{2}+\lambda\frac{L}{2h}\frac{u_s^2}{2}$$

$$\left.\begin{aligned} &[u_{1s}=u_{2s}=u_s] \\ &k=\frac{\Delta p}{L}=\frac{12\eta u_s}{h^2}\end{aligned}\right\}\ \lambda=\frac{\Delta p}{\rho}\cdot\frac{4h}{Lu_s^2}=\frac{12\eta u_s\cdot 4h}{\rho u_s^2 h^2}\rightarrow\boxed{\lambda=\frac{48\nu}{u_s h}=\frac{48}{Re}}$$

*Стационарно, нестишљиво струјање кроз цев



Претпоставке:

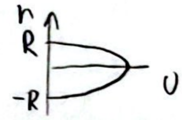
1. Стационарно $\frac{\partial()}{\partial t}=0$
2. Изотермно $T=const.$
3. Нестишљиво $\rho=const.$
4. Осносиметрично $\frac{\partial()}{\partial\theta}=0,\ u_\theta=0$
5. Масене силе се занемарују $f=0$

$$\boxed{\frac{dp}{dz}=\eta\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=-K}$$

• брзина

$$\eta\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=-K,\quad d\left(r\frac{du}{dr}\right)=-\frac{K}{\eta}r\,dr\,/\int,\quad r\frac{du}{dr}=-\frac{K}{\eta}\frac{r^2}{2}+C_1,\quad du=-\frac{K}{2\eta}r\,dr+C_1\frac{dr}{r}\rightarrow$$

$$\boxed{u=-\frac{Kr^2}{4\eta}+C_1\ln r+C_2}\ \Big/\begin{matrix}r=R\\u=0\end{matrix}\Big\}\ C_1=0,\ C_2=\frac{K}{\eta}\frac{R^2}{4}\rightarrow\boxed{u=\frac{K}{4\eta}\left(R^2-r^2\right)}$$



• максимална брзина

$$u'=0\rightarrow\boxed{r=0}\rightarrow\boxed{u_{max}=\frac{R^2K}{4\eta}}$$

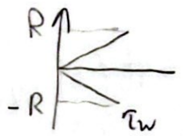
• проток и средња брзина

$$\dot V=2\pi\int_0^R u\,r\,dr=\boxed{\frac{K\pi R^4}{8\eta}}\qquad u_s=\dot V\cdot R^2\pi\rightarrow\boxed{u_s=\frac{KR^2}{8\eta}}$$

• тангенцијални напон

$$\tau=-\eta\frac{du}{dr}=\frac{Kr}{2}$$

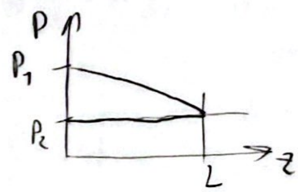
$$\tau_w=-\eta\frac{du}{dr}\Big|_{r=R}=\frac{KR}{2}$$



• расподела притиска

$$\frac{dp}{dz}=-K,\quad\boxed{p=-Kz+C}$$

$$\begin{matrix}z=0, & p=p_1\\ z=L, & p=p_2\end{matrix}\Big\}\begin{matrix}C=p_1\\ p_2=-KL+C\end{matrix}\Big\}\ p_1=p_2+KL\rightarrow\boxed{K=\frac{\Delta p}{L}}\rightarrow\boxed{p=p_1-\frac{\Delta p}{L}z}$$



• коеф. трења

$$\frac{p_1}{\rho}+\frac{u_{1s}^2}{2}=\frac{p_2}{\rho}+\frac{u_{2s}^2}{2}+\lambda\frac{L}{D_h}\frac{u_{2s}^2}{2}$$

$$D_h=\frac{4A}{O}=\frac{4\pi D^2}{4\pi D}=D$$

$$u_{1s}=u_{2s}=u$$

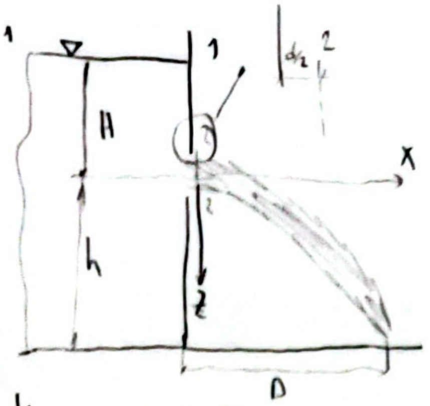
$$K=\frac{8\eta u_s}{R^2}$$

$$\lambda=\frac{\Delta p}{\rho}\cdot\frac{4R}{Lu_s^2}=\frac{8\eta u_s}{R^2}\cdot\frac{4R}{\rho u_s^2}\rightarrow\boxed{\lambda=\frac{32\nu}{Ru_s}=\frac{64\nu}{Du}=\frac{64}{Re}}$$

Истицање

– Стационарно → ниво воде је исти

* Кроз мали отвор



Б. 1-2

$$\frac{p_1}{\rho}+gH=\frac{p_2}{\rho}+\frac{U_2^2}{2}(\zeta+1)\rightarrow \boxed{U_2=\frac{1}{\sqrt{\zeta+1}}\sqrt{2\left(\frac{p_1-p_2}{\rho}+gH\right)}}$$

$$\frac{1}{\sqrt{\zeta+1}}=\varphi$$ – коефицијент брзине

ψ – коеф. контракције

$\boxed{\mu=\psi\cdot\varphi}$ – коеф. истока

– Када је $p_1=p_2=p_a$

$$\boxed{U_2=\varphi\sqrt{2gH}}\qquad\boxed{\dot V=\mu A\sqrt{2gH}}$$

• даље:

$$x:\ \dot x=U_2,\ x=U_2t+C_1\ \Big/\ {}^{t=0}_{x=0},\ C_1=0\rightarrow \boxed{x=U_2t}$$

$$z:\ \ddot z=g,\ \dot z=gt+C_1,\ z=g\frac{t^2}{2}+C_1t+C_2\rightarrow \boxed{z=g\frac{t^2}{2}}$$

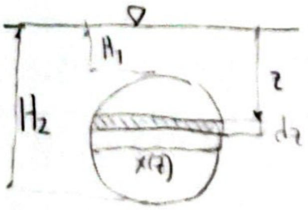
$$\boxed{x=U_2\cdot\sqrt{\frac{2z}{g}}}$$

↳ $z=h\rightarrow x=D=\frac{d}{2}+\varphi\sqrt{2gH}\cdot\sqrt{\frac{2h}{g}}$

$$\boxed{D=\frac{d}{2}+2\varphi\sqrt{Hh}}$$

* Кроз велики отвор

– Брзине струје по висини се немогу занемарити



$$\frac{p_a}{\rho}+gz=\frac{p_a}{\rho}+(1+\zeta)\frac{U_2^2}{2}\rightarrow \boxed{U_2=\varphi\sqrt{2gz}}$$

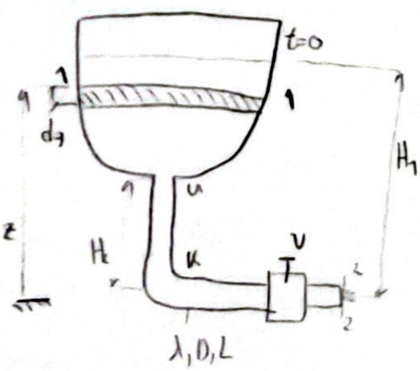
$$d\dot V=U_2(z)\,dA_2,\quad dA_2=\psi\,x(z)\,dz$$

$$\dot V=\int d\dot V=\int_{H_1}^{H_2}\psi\varphi\,x(z)\sqrt{2gz}\,dz\rightarrow \boxed{\dot V=\int_{H_1}^{H_2}\mu\,x(z)\sqrt{2gz}\,dz}$$

* Квазистационарно

– $D\gg d\rightarrow U_1\ll U_2$

↳ квазистационарно



$$\boxed{dV=\dot V(z)\,dt=-A(z)\,dz}\quad (1)$$

$$\frac{p_1}{\rho}+gz=\frac{p_2}{\rho}+\left(\zeta_u+\zeta_k+\zeta_v+1+\lambda\frac{L}{D}\right)\frac{U_2^2}{2}\rightarrow U_2=\varphi\sqrt{2\left(\frac{p_1-p_2}{\rho}+gz\right)}$$

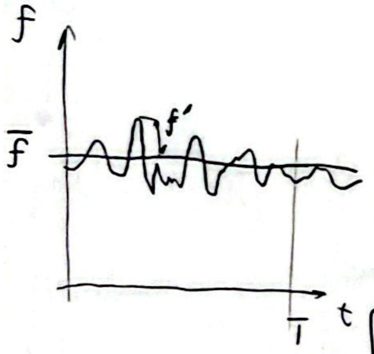
$$\boxed{\dot V=U_2A_2=\mu\frac{d^2\pi}{4}\sqrt{2\left(\frac{p_1-p_2}{\rho}+gz\right)}}=\dot V dt\quad (2)$$

$$(1)=(2)$$

Рејнолдсов број

$$\boxed{Re=\frac{UD}{\nu}},\quad \nu=\frac{\eta}{\rho}$$

- Ламинарно: $Re<2320$ – размена кол. кр. између слојева на микроскопском нивоу
- Турбулентно $Re\geq 2320$ – || – макроскопском нивоу
  ↳ равномернији профили брзине



$$\boxed{\bar{f}(x,y,z)=\frac{1}{T}\int_0^T f(x,y,z,t)\,dt}$$ – средња физ. вел. у једном временском периоду

$$\boxed{f(x,y,z,t)=\bar{f}(x,y,z)+f'(x,y,z,t)} \Rightarrow u=\bar{u}+u'$$

↳ вредност физ. вел. је збир средње вредности и флуктуације

$$\boxed{\bar{f'}(x,y,z)=\frac{1}{T}\int_0^T f'\,dt=0}$$

• Примена Навије-Стоксову ј-ну

$$\rho\frac{D\vec{U}}{Dt}=\rho\vec{f}-\mathrm{grad}\,p+\Delta\vec{U}\eta \rightarrow \frac{1}{T}\int_0^T\rho\frac{D\vec{U}}{Dt}dt=\frac{1}{T}\int_0^T(\rho\vec{f}-\mathrm{grad}\,p+\eta\Delta\vec{U})\;/\vec{i}$$

X: $$\frac{1}{T}\int_0^T\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)dt=\frac{1}{T}\int_0^T\left(\rho f_x-\frac{dp}{dx}+\eta\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)\right)$$

$$\frac{du^2}{dx}=\frac{du}{dx}u+\frac{du}{dx}u \rightarrow \frac{du}{dx}u=\frac{du^2}{dx}-\frac{du}{dx}u\;;\quad \frac{d(uw)}{dz}=u\frac{dw}{dz}+w\frac{du}{dz}\rightarrow w\frac{du}{dz}=\frac{d(uw)}{dz}-u\frac{dw}{dz}$$

$$\frac{d(uv)}{dy}=u\frac{dv}{dy}+v\frac{du}{dy}\rightarrow v\frac{du}{dy}=\frac{d(uv)}{dy}-u\frac{dv}{dy}\;;$$

↓

$$\frac{1}{T}\int_0^T\rho\Big(\underbrace{\frac{du}{dt}}_{I}+\underbrace{\frac{du^2}{dx}-u\frac{du}{dx}+\frac{d(uv)}{dy}-u\frac{dv}{dy}+\frac{d(uw)}{dz}-u\frac{dw}{dz}}_{II}\Big)=\frac{1}{T}\int_0^T\Big(\underbrace{\rho f_x}_{III}-\underbrace{\frac{dp}{dx}}_{IV}+\eta\underbrace{\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)}_{V}$$

- локални део инерцијалне силе: $\frac{1}{T}\int_0^T \rho \frac{\partial u}{\partial t} dt = \rho \frac{\partial}{\partial t}\frac{1}{T}\int_0^T u\, dt = \boxed{\rho \frac{\partial \bar{u}}{\partial t}}$

- конвективни део инерцијалне силе:

$$u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + u\frac{\partial w}{\partial t} = u\,\mathrm{div}\,\vec{v} = \underline{0}$$

$$\frac{1}{T}\int_0^T \rho \frac{\partial u^2}{\partial x} dt = \rho\frac{\partial}{\partial x}\frac{1}{T}\int_0^T \left(\bar{u}^2 + 2\bar{u}\overset{0}{u'} + u'^2\right) dt = \boxed{\rho\frac{\partial}{\partial x}\left(\bar{u}^2 + \overline{u'u'}\right)}$$

$$\frac{1}{T}\int_0^T \rho \frac{\partial (uv)}{\partial y} dt = \rho\frac{\partial}{\partial y}\frac{1}{T}\int_0^T uv\, dt = \rho\frac{\partial}{\partial y}\frac{1}{T}\int_0^T (\bar{u}+u')(\bar{v}+v') = \rho\frac{\partial}{\partial y}\frac{1}{T}\int_0^T \left(\bar{u}\bar{v} + \cancel{\bar{u}v'} + \cancel{u'\bar{v}} + u'v'\right) dt$$

$$= \boxed{\rho\frac{\partial}{\partial y}\left(\bar{u}\bar{v} + \overline{u'v'}\right)}$$

$$\frac{1}{T}\int_0^T \rho \frac{\partial (uw)}{\partial z} dt = \boxed{\rho\frac{\partial}{\partial z}\left(\bar{u}\bar{w} + \overline{u'w'}\right)}$$

III масена сила

$$\frac{1}{T}\int_0^T \rho f_x\, dt = \boxed{\rho \bar{f}_x}$$

IV притисна сила

$$\frac{1}{T}\int \frac{\partial p}{\partial x} dt = \boxed{\frac{\partial \bar{p}}{\partial x}}$$

V вискозна сила

$$\frac{1}{T}\int_0^T \eta \frac{\partial^2 u}{\partial x^2} dt = \boxed{\eta\frac{\partial^2 \bar{u}}{\partial x^2}}$$

$$\boxed{X:\ \rho\left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z}\right) = \rho\bar{f}_x - \frac{\partial \bar{p}}{\partial x} + \eta\left(\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial^2\bar{u}}{\partial y^2} + \frac{\partial^2\bar{u}}{\partial z^2}\right) \underbrace{- \rho\left(\frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z}\right)}_{\text{дифузна инерцијална сила изазвана флуктуацијама}}}$$

- Рејнолдсов тензор напона

$$\tilde{T}^t = \begin{bmatrix} -\rho\overline{u'^2} & -\rho\overline{u'v'} & -\rho\overline{u'w'} \\ -\rho\overline{v'u'} & -\rho\overline{v'^2} & -\rho\overline{v'w'} \\ -\rho\overline{w'u'} & -\rho\overline{w'v'} & -\rho\overline{w'^2} \end{bmatrix}$$

Сличност

Геометријска сличност - $\frac{L}{L_m} = \frac{x}{x_m} = \frac{y}{y_m} = \frac{z}{z_m} = K_L$

Кинематичка сличност $\frac{t}{t_m} = K_t$; $\frac{U}{U_m} = \frac{u}{u_m} = \frac{v}{v_m} = \frac{w}{w_m} = K_u$

Динамичка сличност - односи сила су исти

. Навије Стокс:

$$\frac{D\vec{U}}{Dt} = \vec{f} - \frac{1}{\rho}\,grad\,p + \nu \Delta \vec{U} + \frac{1}{3}\nu\, grad(div\,\vec{U})$$

$\downarrow$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{3}\eta \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}$$

$\downarrow$ На бездимензијски облик се сведе увођењем референтних вредности

$\tilde{u} = \frac{U}{U_r}$; $\tilde{t} = \frac{t}{T_r}$; $\tilde{x} = \frac{x}{L_r}$; $\tilde{p} = \frac{p}{p_r}$; $\tilde{f} = \frac{f}{f_r}$; $\tilde{\rho} = \frac{\rho}{\rho_r}$

$$\frac{\rho_r U_r}{t_r}\tilde{\rho}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\rho_r U_r^2}{L_r}\tilde{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} = \rho_r f_r \tilde{\rho}\tilde{f}_i - \frac{p_r}{L_r}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\eta_r U_r}{L_r^2}\left(\tilde{\eta}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{3}\tilde{\eta}\frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{u}}{\partial \tilde{x}}\right) \Big/ : \frac{\rho_r U_r^2}{L_r}$$

$$\underbrace{\frac{L_r}{t_r U_r}}_{St}\tilde{\rho}\frac{\partial \tilde{u}}{\partial t} + \tilde{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} = \underbrace{\frac{f_r L_r}{U_r^2}}_{\frac{1}{Fr}}\tilde{\rho}\tilde{f} - \underbrace{\frac{p_r}{\rho_r U_r^2}}_{Eu}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \underbrace{\frac{\eta_r}{L_r U_r \rho_r}}_{\frac{1}{Re}}\left(\tilde{\eta}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{3}\tilde{\eta}\frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{u}}{\partial \tilde{x}}\right)$$

1. Струхалов број $\boxed{St = \frac{L_r}{t_r U_r}}$
2. Фрудов број $\boxed{Fr = \frac{U_r^2}{f_r L_r}}$
3. Ојлеров број $\boxed{Eu = \frac{p_r}{\rho_r U_r^2}}$
4. Рејнолдсов број $\boxed{Re = \frac{U_r L_r \rho_r}{\eta_r}}$

*Димензиска анализа

$[\tau_w] = \frac{M}{T^2 L}$; $[D] = L$; $[\rho] = \frac{M}{L^3}$; $[\eta] = \frac{M}{LT}$; $[U] = \frac{L}{T}$; $[\delta] = L$

а) $\Pi_{\tau_w} = D^x \rho^y U^z \tau_w$

$M^0 L^0 T^0 = L^x \left(\frac{M}{L^3}\right)^y \left(\frac{L}{T}\right)^z \frac{M}{T^2 L}$

$\left\{ \begin{array}{l} 0 = y+1 \\ 0 = x-3y+z-1 \\ 0 = -z-2 \end{array} \right\} \left\{ \begin{array}{l} y=-1 \\ z=-2 \\ x=0 \end{array} \right\} \boxed{\Pi_{\tau_w} = \frac{\tau_w}{\rho U^2}}$

б) $\Pi_{\eta} = D^x \rho^y U^z \eta$

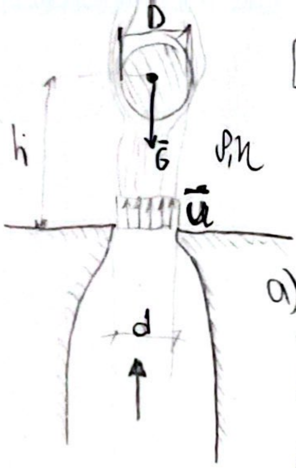
$\boxed{\Pi_{\eta} = \frac{1}{Re}}$

в) $\Pi_{\delta} = D^x \rho^y U^z \delta \rightarrow \boxed{\Pi_{\delta} = \frac{\delta}{D}}$

$\phi(\Pi_{\tau_w}, \Pi_{\eta}, \Pi_{\delta}) = 0 \rightarrow \phi\left(\frac{\tau_w}{\rho U^2}, \frac{1}{Re}, \frac{\delta}{D}\right) = 0 \rightarrow \boxed{\tau_w = f(Re, K) \frac{\rho U^2}{2}}$

$\boxed{Y_{\lambda} = \lambda \frac{L}{D} \frac{U^2}{2} \ ; \ \lambda = 4f}$

②



$[U]=\frac{L}{T}$; $[d]=L$; $[\rho]=\frac{M}{L^3}$; $[D]=L$; $[\eta]=\frac{M}{LT}$; $[G]=\frac{ML}{T^2}$ $[h]=L$

- Ocnalre leuuune: ρ, U, D

$G=m\cdot g=M\cdot L\,T^{-2}$

a) 1° $\pi_d=\rho^x U^y D^z d$

$M^0L^0T^0=(ML^{-3})^x\cdot(LT^{-1})^y\cdot L^z\cdot L$

$M^0L^0T^0=M^x\cdot L^{-3x}\cdot L^y\cdot T^{-y}\cdot L^z\cdot L$

$$\begin{cases}0=x\\0=-3x+y+z+1\\0=-y\end{cases}\ \begin{cases}\boxed{x=0}\\\boxed{z=-1}\\\boxed{y=0}\end{cases}\ \boxed{\pi_d=\frac{d}{D}}$$

2°

$\pi_\eta=\rho^x U^y D^z \eta$

$M^0L^0T^0=M^xL^{-3x}\cdot L^yT^{-y}\cdot L^z\cdot ML^{-1}T^{-1}$

$$\begin{cases}0=x+1\\0=-3x+y+z-1\\0=-y-1\end{cases}\ \begin{cases}\boxed{x=-1}\\\boxed{z=-1}\\\boxed{y=-1}\end{cases}\ \boxed{\pi_\eta=\frac{\eta}{\rho U D}=\frac{1}{Re}}$$

3° $\pi_G=\rho^x U^y D^z G$

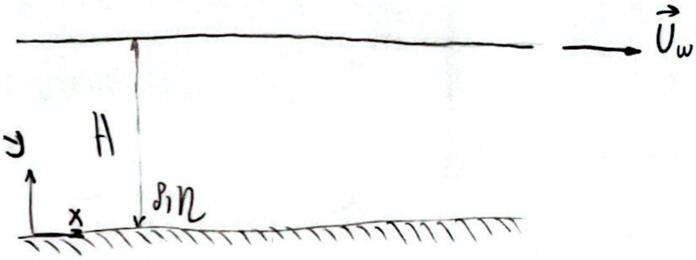
$M^0L^0T^0=M^xL^{-3x}\cdot L^yT^{-y}\cdot L^z\cdot MLT^{-2}$

$$\begin{cases}0=x+1\\0=-3x+y+z+1\\0=-y-2\end{cases}\ \begin{cases}\boxed{x=-1}\\\boxed{z=-2}\\\boxed{y=-2}\end{cases}\ \boxed{\pi_G=\frac{G}{\rho U^2D^2}}$$

4°

1.

$$\tau_w = f\frac{\rho U_{sr}^2}{2}, \quad \lambda = 4f$$



$$[\rho] = \frac{M}{L^3}; \quad [U] = \frac{L}{T}; \quad [D] = L$$

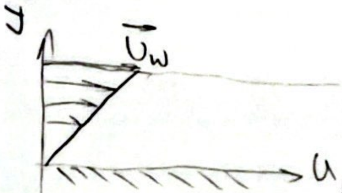
$$[\sigma] = L; \quad [\eta] = \frac{M}{LT}; \quad [\tau_w] = \frac{M}{LT^2}$$

$$\Delta p = 0 \rightarrow \boxed{\frac{dp}{dx} = 0}, \quad \rho = const, \quad \frac{d()}{dt} = 0$$

• брзина u

$$\frac{d^2u}{dy^2} = 0, \quad \frac{du}{dy} = C_1, \quad \boxed{u = C_1 y + C_2}$$

$$\left\{\begin{array}{ll} y=0 & u=0 \\ y=H & u=U_w \end{array}\right. \quad \boxed{C_2 = 0}, \quad U_w = C_1 H \rightarrow \boxed{C_1 = \frac{U_w}{H}} \quad \boxed{u(y) = \frac{U_w}{H} y}$$

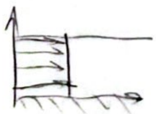


• Проток и средња брзина

$$\dot{V} = \int_0^H u\,dy = \frac{U_w}{H}\frac{y^2}{2}\Big|_0^H = \boxed{\frac{U_w H}{2}}; \quad \dot{V} = U_{sr} \cdot H \rightarrow \boxed{U_{sr} = \frac{U_w}{2}}$$

• Тангенцијални напон

$$\tau = +\eta\frac{du}{dy} = +\boxed{\eta\frac{U_w}{H}} \quad \boxed{\tau = \tau_w = const.}$$



• Димензиона анализа

$$\frac{M}{LT^2} = \frac{f}{2} \cdot \frac{M}{L^3} \cdot \frac{L^2}{T^2} \rightarrow \quad f \text{ - бездимензиона} \rightarrow [f] = M^0 L^0 T^0$$

$$\tau_w = \frac{\lambda}{4}\frac{\rho U_{sr}^2}{2} \rightarrow \boxed{\lambda = \frac{8\tau_w}{\rho U_{sr}^2}}$$